

Linear Differential Invariance Under an Operator Related to the Laplace Transformation

A Dissertation

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN
THE UNIVERSITY OF MICHIGAN

BY

EARL D. RAINVILLE

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LINEAR DIFFERENTIAL INVARIANCE UNDER AN OPERATOR RELATED TO THE LAPLACE TRANSFORMATION.*¹

By EARL D. RAINVILLE.

1. Introduction. The Laplace integral transformation²

$$(1) \quad \mathcal{L}\{F(t)\} \equiv \int_0^{\infty} e^{-st} F(t) dt = f(s),$$

is one which associates with each function $F(t)$ of sufficient regularity another function $f(s)$. Elementary known³ properties of the operator $\mathcal{L}$ include

$$(2) \quad \mathcal{L}\left\{\frac{d^n F(t)}{dt^n}\right\} = s^n f(s) - \sum_{k=0}^{n-1} s^{n-1-k} \left(\frac{d^k F}{dt^k}\right)_{t=0},$$

and

$$(3) \quad \mathcal{L}\{t^k F(t)\} = (-1)^k \frac{d^k}{ds^k} f(s).$$

The Laplace transformation has important applications⁴ to the solution of boundary value problems in ordinary and partial linear differential equations. The operator $\mathcal{L}$ often transforms one differential equation into another which is more readily solved, one which, indeed, may even be algebraic. The transformed equation may be of higher order, or otherwise more complicated, than the original. Finally, we see that many equations do not change form in any essential way when subjected to the operator $\mathcal{L}$. One such equation is

$$(4) \quad \frac{d^2 F}{dt^2} + t^2 F = 0,$$

* Received August 15, 1939.

¹ Presented to the Society Nov. 26, 1938 under a slightly different title.

² For an extensive treatment of this transformation see G. Doetsch, *Theorie und Anwendung der Laplace-Transformation*, Berlin, 1937.

³ Enzo Levi has shown that the case $n=1$ of equation (2) above, together with certain conditions on $F(t)$ and its transform is sufficient to characterize the operator completely. For the precise result see his paper, "Proprietà caratteristiche della trasformazione di Laplace," *Rend. Accad. Lincei*, (6), vol. 24 (1936), pp. 422-426.

⁴ See, for example, R. V. Churchill, "The solution of linear boundary value problems in physics by means of the Laplace transformation": I, *Mathematische Annalen*, vol. 114 (1937), pp. 591-613; II, *Mathematische Annalen*, vol. 115 (1938), pp. 720-739. See also his paper, "On the problem of temperatures in a non-homogeneous bar with discontinuous initial temperatures," *American Journal of Mathematics*, vol. 61 (1939), pp. 651-664, in which the Laplace transformation is used to establish a uniqueness theorem.

for which the transformed equation is

$$(5) \quad \frac{d^2 f}{ds^2} + s^2 f = s(F)_{t=0} + \left(\frac{dF}{dt} \right)_{t=0},$$

as may be seen from (2) and (3).

Our fundamental problem is suggested by the fact that (4) is essentially invariant under $\mathcal{L}$. In order to use only those properties of $\mathcal{L}$ which are concerned in that invariance we introduce another operator σ and study σ instead of $\mathcal{L}$.

DEFINITION 1. Let $D \equiv d/dx$ be the usual symbol for differentiation with respect to x ; let $D^0 \equiv 1$. Then any polynomial in D and x will be called a *linear differential operator of type P*.

DEFINITION 2. Let k, n, k_s, n_s ; $s = 1, 2, \dots$, be non-negative integers. We define σ as a linear operator on linear differential operators of type P by

$$(6) \quad \sigma x^k D^n = (-1)^k D^k x^n,$$

and

$$(7) \quad \sigma \left[\sum_s a_s x^{k_s} D^{n_s} \right] = \sum_s a_s \sigma (x^{k_s} D^{n_s}),$$

where the a_s are any constants.

It should be noted that by $D^k x^n$ we mean that differential operator which, acting upon a function F , yields the k -th derivative of the product $x^n F$.

It is of some value to keep in mind one aspect of the nature of σ . Consider a given function $f(x)$, taken to be single valued for the present purpose. We may associate with $f(x)$ an operator f which transforms each number x of a certain set of numbers into another number $f(x)$ in another, or the same, set of numbers. We call f an operator of class one. Next consider D . The operator D transforms each function of a certain set of functions into another function of x . Further, if D operate on numbers, the result is trivial; i. e., D transforms every number into the same number, zero. Hence, we call D an operator of class two, noting that in a sense D must operate on operators of class one to give non-trivial results. Now consider σ . This operator is defined above in such a way that it transforms each linear differential operator into another linear differential operator. Essentially σ needs to operate on operators of class two to give non-trivial results. We call σ an operator of class three.

An adjoint operator may be defined such that it changes a linear dif-

⁵ Essentially this definition is to be found in S. Pincherle and U. Amaldi, *Operazioni distributive*, Bologna, 1901, p. 361.

ferential operator, not necessarily of type P , into its adjoint linear differential operator. This adjoint operator ⁶ is of class three in the above sense.

2. Results. Some useful, not all new, properties of σ are obtained. Two linear bases are found for the set of linear differential operators invariant under σ . Two invariant second order differential operators are found to form a fundamental system of invariant operators; i. e., any invariant operator may be expressed as a polynomial in these two operators. A linear basis is exhibited for what are called t -variants (Definition 3) with respect to σ .

Linear operational equations in σ are completely solved in the case of constant coefficients. This is done with the aid of two theorems on the representation of linear differential operators in terms of t -variants or of invariants and pseudo-invariants. The same tools are useful in the solution of linear operational equations in σ with variable (linear differential operational) coefficients, as is demonstrated in the example worked out in Section 10.

In Section 9 certain results are specialized to yield a classification of the differential equations, such as (4) above, invariant under σ .

3. Preliminary definitions and lemmas. Since the only linear differential operators to enter this study are of type P , we shall often hereafter omit mention of this restriction.

DEFINITION 3. If y is a linear differential operator such that $\sigma y = ty$ where $t^4 = 1$, then y will be called a *linear differential t -variant* with respect to σ . A 1-variant will be referred to on occasion as an *invariant* and a (-1)-variant may be called a *pseudo-invariant*.

DEFINITION 4. The *degree* and the *order* of a linear differential operator are respectively the highest power of the independent variable and the order of the highest ordered derivative appearing explicitly in the operator.

LEMMA 1. If y is a linear differential operator, then the degree of $\sigma y =$ the order of y and the order of $\sigma y =$ the degree of y .

This lemma is an immediate consequence of the definition of σ . The application of Lemma 1 leads to

⁶ E. D. Rainville, "Adjoints of linear differential operators," *American Mathematical Monthly*, vol. 46 (1939), pp. 623-627. For relations between σ and the adjoint operator, see L. Schlesinger, *Handbuch der Linearen Differentialgleichungen*, Leipzig, 1895, vol. 1, p. 426 and E. D. Rainville, "A discrete group arising in the study of differential operators," as yet unpublished.

LEMMA 2. For a linear differential operator to be t -variant with respect to σ it is necessary that its degree equal its order.

LEMMA 3. The linear differential operators $A_1 = D^2 + x^2$ and $A_2 = x^2 D^2 + 2xD$, are invariant with respect to σ .

We prove Lemma 3 by direct evaluation of σA_1 and σA_2 .

$$\begin{aligned}\sigma A_1 &= x^2 + D^2 = A_1, \\ \sigma A_2 &= D^2 x^2 - 2Dx = x^2 D^2 + 4xD + 2 - 2xD - 2 = A_2.\end{aligned}$$

THEOREM 1. If A and B are linear differential operators, then

$$\sigma(AB) = (\sigma A)(\sigma B).$$

Let $v = x^k D^n$, then

$$\sigma(xv) = \sigma(x^{k+1} D^n) = (-1)^{k+1} D^{k+1} x^n = -D\sigma(x^k D^n),$$

so that we have

$$(8) \quad \sigma(xv) = (\sigma x)(\sigma v).$$

Next,

$$\begin{aligned}\sigma(Dv) &= \sigma(x^k D^{n+1} + kx^{k-1} D^n) = (-1)^k D^k x^{n+1} + (-1)^{k-1} k D^{k-1} x^n \\ &= (-1)^k x D^k x^n + (-1)^k k D^{k-1} x^n + (-1)^{k-1} k D^{k-1} x^n = (-1)^k x D^k x^n = x(\sigma v)\end{aligned}$$

Then

$$(9) \quad \sigma(Dv) = (\sigma D)(\sigma v).$$

Theorem 1 follows directly from (8) and (9). Further,

$$(10) \quad \sigma^2(AB) = \sigma[(\sigma A)(\sigma B)] = (\sigma^2 A)(\sigma^2 B),$$

and, for any integral $k \geq 0$, $\sigma^k(AB) = (\sigma^k A)(\sigma^k B)$.

LEMMA 4. If $v = x^k D^n$, then $\sigma^2 v = (-1)^{k+n} v$.

By (10) above

$$\sigma^2 v = (\sigma^2 x^k)(\sigma^2 D^n) = [\sigma(-1)^k D^k][\sigma x^n] = (-1)^k x^k (-1)^n D^n = (-1)^{k+n} v.$$

Lemma 4 itself leads at once to ⁷

THEOREM 2. If y is a linear differential operator, then $\sigma^4 y = y$.

4. First classification of invariants. Direct application of Theorem 1 and Lemma 3 yields

THEOREM 3. Any linear combination of terms of the type

$$A_1^{m_1} A_2^{m_2} A_1^{m_3} \cdots A_1^{m_{s-1}} A_2^{m_s}$$

⁷ Theorem 2 appears in Pincherle and Amaldi, *loc. cit.*, p. 357.

in which $m_i; i = 1, 2, \dots, s$, are non-negative integers, is a linear differential invariant with respect to σ .

Next we obtain a simple necessary condition for invariance under σ .

LEMMA 5. A necessary and sufficient condition that a linear differential operator be invariant under σ^2 is that, for each term $a_{kn}x^kD^n$ of the operator, $k \equiv n \pmod{2}$.

This follows at once from Lemma 4. Noting that invariance under σ implies invariance under σ^2 , we have a necessary condition for the former.

THEOREM 4. For each term $a_{kn}x^kD^n$ of a linear differential invariant with respect to σ it is true that $k \equiv n \pmod{2}$.

DEFINITION 5. The leading term of a linear differential operator is the non-vanishing term of highest degree among those terms of highest order in the operator.

It will prove useful to note that, since the order of the operator is the order of its leading term, we have

LEMMA 6. In a linear differential t -variant the degree of the leading term does not exceed its order.

DEFINITION 6. By linear differential invariants of type H we mean the set of invariant operators

$$(11) \quad A_1^{n-k}A_2^k; \quad 0 \leq k \leq n,$$

and

$$(12) \quad \frac{1}{4(n-k)(k+1)} [A_1^{n-k}A_2^{k+1} - A_2^{k+1}A_1^{n-k}]; \quad 0 \leq k < n.$$

LEMMA 7. The leading term of $A_1^{n-k}A_2^k; 0 \leq k \leq n$, is $x^{2k}D^{2n}$.

This follows at once from the definitions of A_1 and A_2 .

LEMMA 8. The leading term of

$$\frac{1}{4(n-k)(k+1)} [A_1^{n-k}A_2^{k+1} - A_2^{k+1}A_1^{n-k}]; \quad 0 \leq k < n,$$

is $x^{2k+1}D^{2n+1}$.

In the proof of Lemma 8 we shall use the convention that

$$y = a_{\delta\epsilon}x^\delta D^\epsilon + a_{\delta'\epsilon'}x^{\delta'}D^{\epsilon'} + \dots$$

means that y is a linear differential operator with leading term $a_{\delta\epsilon}x^\delta D^\epsilon$ and that the leading term of $(y - a_{\delta\epsilon}x^\delta D^\epsilon)$ is $a_{\delta'\epsilon'}x^{\delta'}D^{\epsilon'}$.

With the above convention note that the formula

$$(13) \quad A_2^k = x^{2k}D^{2k} + 2k^2x^{2k-1}D^{2k-1} + \dots$$

holds for $k=1$ and in a trivial sense for $k=0$. Assume (13) to hold for some k . Then

$$\begin{aligned} A_2^{k+1} &= x^{2k+2}D^{2k+2} + (4k + 2k^2 + 2)x^{2k+1}D^{2k+1} + \dots \\ &= x^{2k+2}D^{2k+2} + 2(k+1)^2x^{2k+1}D^{2k+1} + \dots, \end{aligned}$$

and it follows by induction that (13) is true for any $k \geq 0$. Now

$$(14) \quad A_1^{n-k}A_2^k = x^{2k}D^{2n} + 2k(2n-k)x^{2k-1}D^{2n-1} + \dots$$

holds for $n=k \geq 0$. Assume (14) to hold for some pair of numbers n, k . Then

$$\begin{aligned} (15) \quad A_1^{n-k+1}A_2^k &= x^{2k}D^{2n+2} + [4k + 2k(2n-k)]x^{2k-1}D^{2n+1} + \dots \\ &= x^{2k}D^{2n+2} + 2k[2(n+1)-k]x^{2k-1}D^{2n+1} + \dots, \end{aligned}$$

so that by induction (14) holds for any $n \geq k \geq 0$.

In view of (13) the application of A_2^k to A_1^{n-k+1} is seen to yield

$$(16) \quad A_2^kA_1^{n-k+1} = x^{2k}D^{2n+2} + 2k^2x^{2k-1}D^{2n+1} + \dots,$$

for $n \geq k \geq 0$. Combining (15) and (16) with k replaced by $(k+1)$, we have as the leading term of $[A_1^{n-k}A_2^{k+1} - A_2^{k+1}A_1^{n-k}]$; $0 \leq k < n$, the expression $4(k+1)(n-k)x^{2k+1}D^{2n+1}$, so that Lemma 8 is established.

THEOREM 5. *A necessary and sufficient condition that there exist a linear differential invariant with respect to σ with leading term $x^\epsilon D^\epsilon$ is that either*

$$\delta \equiv \epsilon \equiv 0 \pmod{2}, \quad 0 \leq \delta \leq \epsilon,$$

or

$$\delta \equiv \epsilon \equiv 1 \pmod{2}, \quad 1 \leq \delta < \epsilon.$$

Lemmas 7 and 8 exhibit linear differential invariants for each leading term indicated in Theorem 5. We proceed to show that no linear differential invariant can exist with leading term not proportional to one of those indicated in Theorem 5. By Theorem 4 we must have $\delta \equiv \epsilon \pmod{2}$. By Lemma 6 we must have $\delta \leq \epsilon$. We have left the one case $\delta = \epsilon = 2h+1$ and we consider that now. If a linear differential operator y had for its leading term $x^{2h+1}D^{2h+1}$, then σy would have for its leading term $(-x^{2h+1}D^{2h+1})$, and y could not be invariant. This concludes the proof of Theorem 5. We proceed to the main result of this section.

THEOREM 6. *Any linear differential invariant with respect to σ is a linear combination of linear differential invariants of type H.*

Stated in another way, we show that the invariants of type H form a linear (infinite) basis for the algebra whose elements are the invariants with respect to σ .

Let y be any linear differential invariant with leading term $a_{\delta\epsilon}x^\delta D^\epsilon$, where, of course, δ and ϵ are subject to the restrictions of Theorem 5. By Lemmas 7 and 8 there exists a linear differential invariant of type H with leading term $x^\delta D^\epsilon$. Hence we see that there exists a linear combination of y and a linear differential invariant of type H (with coefficient of y not zero) which is invariant under σ and is such that its leading term is either (a) of lower order than the leading term of y , or (b) of the same order and of lower degree than the leading term of y . Repetition of this argument shows that there exists an identically vanishing linear combination (with coefficient of y not zero) of y and linear differential invariants of type H . Thus Theorem 6 is established.

Next we note that, since no two of the linear differential invariants of type H have proportional leading terms, it follows that the linear differential invariants of type H are linearly independent.

The preceding work, particularly Theorem 6, shows that A_1 and A_2 form a fundamental system of invariant differential operators in the sense of

THEOREM 7. *Any linear differential operator invariant with respect to σ may be expressed as a polynomial in A_1 and A_2 .*

Of course, Theorem 3 has already stated that any polynomial in A_1 and A_2 is invariant under σ . Since A_1 is not commutative with A_2 , the word polynomial is used here in the sense of linear combinations of operators of the type exhibited in Theorem 3.

5. Second classification of invariants.

LEMMA 9. *A necessary and sufficient condition for*

$$I_{kn} = x^k D^n + \sigma(x^k D^n); \quad 0 \leq k, n,$$

to be a linear differential invariant with respect to σ is that $k \equiv n \pmod{2}$.

Noting that

$$\sigma I_{kn} = \sigma(x^k D^n) + \sigma^2(x^k D^n),$$

and recalling Lemma 4 we see that Lemma 9 follows at once.

DEFINITION 7. By *linear differential invariants of type J* we mean the set of invariant operators

$$\begin{aligned} &I_{kn}; \quad k \equiv n \equiv 0 \pmod{2}, \quad 0 \leq k \leq n, \\ \text{and} \quad &I_{kn}; \quad k \equiv n \equiv 1 \pmod{2}, \quad 1 \leq k < n. \end{aligned}$$

Note that in the set of linear differential invariants of type J whenever $k \neq n$ the leading term of I_{kn} is $x^k D^n$; if $k = n$, then k and n are even and the leading term of I_{nn} is $2x^n D^n$. Hence it is evident that the linear differential invariants of type J are linearly independent.

Since all leading terms permitted by Theorem 5 are included in type J , we may follow the line of reasoning used to prove Theorem 6 and thus demonstrate

THEOREM 8. *Any linear differential operator invariant with respect to σ may be expressed linearly in terms of linear differential invariants of type J .*

See also the remark directly below Theorem 6.

6. A classification of t -variants. We shall briefly indicate a classification of t -variants similar to the above second classification of invariants. This done, we may consider t -variants completely specified and may proceed to two representation theorems with the aid of which we solve linear operational equations in σ .

From Lemma 4 of Section 3 we get

LEMMA 10. *A necessary and sufficient condition that a linear differential operator be pseudo-invariant with respect to σ^2 is that, for each term $a_{kn}x^k D^n$ of the operator, $k \equiv n + 1 \pmod{2}$.*

Let $i = \sqrt{-1}$. If $t = i$ or if $t = i^3$, then $t^2 = -1$ and any corresponding linear differential t -variant with respect to σ is pseudo-invariant with respect to σ^2 . If $t = -1$, then we have actual invariance with respect to σ^2 . Hence Lemmas 5 and 10 lead to

THEOREM 9. *For each term $a_{kn}x^k D^n$ of a linear differential t -variant with respect to σ it is true that $k \equiv n + \frac{1}{2}(1 - t^2) \pmod{2}$.*

LEMMA 11. *A necessary and sufficient condition that*

$$I_{kn}^{(t)} = x^k D^n + t^3 \sigma(x^k D^n), \quad t^4 = 1,$$

be a t -variant with respect to σ is that $k \equiv n + \frac{1}{2}(1 - t^2) \pmod{2}$.

Since

$$\sigma I_{kn}^{(t)} = \sigma(x^k D^n) + t^3 \sigma^2(x^k D^n)$$

and

$$t I_{kn}^{(t)} = t x^k D^n + \sigma(x^k D^n),$$

a necessary and sufficient condition for the equality of $\sigma I_{kn}^{(t)}$ and $t I_{kn}^{(t)}$ is that

$\sigma^2(x^k D^n) = t^2 x^k D^n$. By Lemma 4 this is equivalent to $t^2 = (-1)^{k+n}$ or to $k \equiv n + \frac{1}{2}(1 - t^2) \pmod{2}$.

Considerations similar to those used in the proof of Theorem 5 yield a proof (omitted here) of

THEOREM 10. *A necessary and sufficient condition that there exist a linear differential t -variant with respect to σ with leading term $x^\delta D^\epsilon$ is that either*

$$\begin{aligned} \delta &\equiv \epsilon + \frac{1}{2}(1 - t^2) \equiv 0 \pmod{2}, & 0 \leq \delta < \epsilon + \frac{1}{4}(1 + t^2)(1 + t), \\ \text{or} & & \\ \delta &\equiv \epsilon + \frac{1}{2}(1 - t^2) \equiv 1 \pmod{2}, & 1 \leq \delta < \epsilon + \frac{1}{4}(1 + t^2)(1 - t). \end{aligned}$$

DEFINITION 8. By *linear differential t -variants of type J* we mean the set of t -variant operators

$$\begin{aligned} I_{kn}^{(t)}; \quad k &\equiv n + \frac{1}{2}(1 - t^2) \equiv 0 \pmod{2}, \quad 0 \leq k < n + \frac{1}{4}(1 + t^2)(1 + t), \\ \text{and} & \\ I_{kn}^{(t)}; \quad k &\equiv n + \frac{1}{2}(1 - t^2) \equiv 1 \pmod{2}, \quad 1 \leq k < n + \frac{1}{4}(1 + t^2)(1 - t). \end{aligned}$$

It can be seen that the linear differential t -variants of type J are linearly independent. Reasoning parallel to that used to prove Theorem 6 will demonstrate

THEOREM 11. *Any linear differential t -variant with respect to σ may be expressed linearly in terms of linear differential t -variants of type J .*

See also the remark directly below Theorem 6.

7. Representation theorems. We shall prove the following two theorems on the representation of linear differential operators of type P .

THEOREM 12. *Any linear differential operator of type P may be represented in one, and only one, way in the form*

$$(17) \quad I + P + Q + W$$

where I is an invariant, P a pseudo-invariant, Q an i -variant, and W an i^3 -variant.

THEOREM 13. *Any linear differential operator of type P may be represented in one, and only one, way in the form*

$$(18) \quad I_1 + P_1 + x(I_2 + P_2) + D(I_3 + P_3)$$

where I_1, I_2, I_3 are invariants and P_1, P_2, P_3 are pseudo-invariants.

In order to picture more clearly the relation between Theorems 12 and 13, let us consider the operator $2x^2D$. We may write

$$2x^2D = (-ixD^2 + x^2D - 2iD) + (ixD^2 + x^2D + 2iD),$$

where $Q = -ixD^2 + x^2D - 2iD$ and $W = ixD^2 + x^2D + 2iD$ are respectively an i -variant and an i^3 -variant. Here, though the operator $2x^2D$ is real, the representation (17) introduces the imaginary unit. We may, on the other hand, write

$$2x^2D = x[(-1) + (2xD + 1)],$$

where $I_2 = -1$ and $P_2 = 2xD + 1$ are respectively an invariant and a pseudo-invariant. Hence, using (18) the representation of $2x^2D$ is "real." The representations (17) and (18) play roles corresponding to the two solutions $F = a_1e^{ix} + a_2e^{-ix}$ and $F = c_1 \cos x + c_2 \sin x$ of the differential equation $(D^2 + 1)F = 0$.

The example in Section 10 illustrates the fact that (18) may on occasion have considerable advantage over the apparently simpler and more natural representation (17).

The representation (17) is essentially a result of the fact that σ satisfies the operational equation $\sigma^4 = E$, the identity.

Proof of Theorem 12. First we give an explicit expression for any term $x^k D^n$ of a linear differential operator in the manner desired. Let k, n be non-negative integers. Then, using the notation of Lemma 11, we have

$$(19) \quad x^k D^n = \frac{1}{4}[1 + (-1)^{k+n}][I_{kn}^{(1)} + I_{kn}^{(-1)}] + \frac{1}{4}[1 - (-1)^{k+n}][I_{kn}^{(i)} + I_{kn}^{(-i)}].$$

Because of the linearity of the operators uniqueness of (17) will follow if we show that

$$(20) \quad I + P + Q + W = 0,$$

with the notation of Theorem 12, implies $I = P = Q = W = 0$.

If we operate on (20) with σ^2 we find

$$(21) \quad I + P - Q - W = 0.$$

From (20) and (21) we have $I + P = 0$. Operating on this with σ , we get $I - P = 0$. Hence $I = P = 0$. But (20) and (21) also lead to $Q + W = 0$, from which it follows that $iQ - iW = 0$. Hence $Q = W = 0$ and the proof of Theorem 12 is complete.

Proof of Theorem 13. In order to show the existence of the representation (18) we write the identity, for k and n non-negative integers,

$$(22) \quad \begin{aligned} x^k D^n = & \frac{1}{4} [1 + (-1)^{k+n}] [I_{kn}^{(1)} + I_{kn}^{(-1)}] \\ & + \frac{1}{4} (\operatorname{sgn} k) [1 - (-1)^{k+n}] x [I_{k-1,n}^{(1)} + I_{k-1,n}^{(-1)}] \\ & + \frac{1}{4} (1 - \operatorname{sgn} k) [1 - (-1)^n] D [I_{0,n-1}^{(1)} + I_{0,n-1}^{(-1)}], \end{aligned}$$

in which $\operatorname{sgn} k$ is the usual signum function with argument k .

In order to prove the uniqueness of the representation (18), we may follow in part the method used in the proof of Theorem 12. Assume

$$(23) \quad I_1 + P_1 + x(I_2 + P_2) + D(I_3 + P_3) = 0,$$

with the notation as in Theorem 13. Operating on (23) with σ^2 we find

$$(24) \quad I_1 + P_1 - x(I_2 + P_2) - D(I_3 + P_3) = 0.$$

From these two equations $I_1 + P_1 = 0$ and hence $I_1 = P_1 = 0$ follow. We are left with

$$(25) \quad x(I_2 + P_2) + D(I_3 + P_3) = 0.$$

Here the method of proof digresses from that above. We recall that the degree of a linear differential invariant equals its order and that the same is true of a pseudo-invariant (Lemma 2). Hence the degree of $(I_2 + P_2)$ equals its order, say n_2 . Also the degree of $(I_3 + P_3)$ equals its order, say n_3 . Since the degrees and the orders of the two terms of (25) must be respectively equal, we have $n_2 + 1 = n_3$ and $n_2 = n_3 + 1$. But no pair of values n_2 and n_3 can satisfy these relations. Thus we have $x(I_2 + P_2) = 0$ and $D(I_3 + P_3) = 0$. It follows at once that $I_2 = P_2 = I_3 = P_3 = 0$.

Suppose an operator y is represented in each of the forms (17) and (18). It is then easy to obtain I , P , Q , and W in terms of I_1 , I_2 , I_3 , P_1 , P_2 , P_3 . We find $I = I_1$, $P = P_1$,

$$(26) \quad Q = \frac{1}{2}(D - ix)(I_3 + iI_2) + \frac{1}{2}(D + ix)(P_3 - iP_2),$$

and

$$(27) \quad W = \frac{1}{2}(D + ix)(I_3 - iI_2) + \frac{1}{2}(D - ix)(P_3 + iP_2).$$

An examination of (20) and (22) leads us to believe that the determination of I_2 , I_3 , P_2 , P_3 in terms of Q and W is a term by term affair not to be written in such simple forms as (26) and (27).

8. **Linear operational equations in σ : constant coefficients.** Next we consider linear operational equations in σ in analogy to linear differential equations in D . The unknown to be determined is now a linear differential operator. Since $\sigma^4 y = y$ for y any linear differential operator, we consider only operational equations of "order" ≤ 3 in σ .

First we treat the homogeneous equation

$$(28) \quad a_3 \sigma^3 y + a_2 \sigma^2 y + a_1 \sigma y + a_0 y = 0,$$

where a_3, a_2, a_1, a_0 are constants. We may operate on (28) with σ and obtain

$$a_2 \sigma^3 y + a_1 \sigma^2 y + a_0 \sigma y + a_3 y = 0,$$

$$a_1 \sigma^3 y + a_0 \sigma^2 y + a_3 \sigma y + a_2 y = 0,$$

$$a_0 \sigma^3 y + a_3 \sigma^2 y + a_2 \sigma y + a_1 y = 0.$$

For (28) to have a solution other than $y = 0$ it is necessary that the determinant of the coefficients of $y, \sigma y, \sigma^2 y, \sigma^3 y$, in the above equations vanish. This leads at once to

THEOREM 14. *A necessary condition that there exist a non-vanishing linear differential operator y which satisfies*

$$(28) \quad a_3 \sigma^3 y + a_2 \sigma^2 y + a_1 \sigma y + a_0 y = 0,$$

where a_3, a_2, a_1, a_0 are constants, is that

$$(29) \quad (a_3 + a_2 + a_1 + a_0)(a_3 - a_2 + a_1 - a_0)[(a_3 - a_1)^2 + (a_2 - a_0)^2] = 0.$$

If we now put $y = I + P + Q + W$, Equation (28) yields

$$(30) \quad \begin{aligned} (a_3 + a_2 + a_1 + a_0)I &= \Delta_1 I = 0, \\ (a_3 - a_2 + a_1 - a_0)P &= \Delta_2 P = 0, \\ [(a_3 - a_1) - i(a_2 - a_0)]Q &= \Delta_3 Q = 0, \\ [(a_3 - a_1) + i(a_2 - a_0)]W &= \Delta_4 W = 0, \end{aligned}$$

with the aid of Theorem 12. By means of Equations (30) we are able to determine the general solution of (28). For example, if $\Delta_1 = \Delta_3 = \Delta_4 = 0$ and $\Delta_2 \neq 0$, then the general solution of (28) is $y = I + Q + W$, where I is any invariant, Q any i -variant and W is any i^3 -variant.

We turn now to the non-homogeneous case.

THEOREM 15. *Let a_3, a_2, a_1, a_0 be constants such that $\Delta_1 \Delta_2 \Delta_3 \Delta_4 \neq 0$, where the Δ 's are as defined in (30). Let I, P, Q, W be as defined in Theorem*

12. Then there exists one, and only one, linear differential operator which satisfies the equation

$$(31) \quad a_3 \sigma^3 y + a_2 \sigma^2 y + a_1 \sigma y + a_0 y = I + P + Q + W,$$

namely

$$(32) \quad y = \frac{I}{\Delta_1} - \frac{P}{\Delta_2} + i \frac{Q}{\Delta_3} - i \frac{W}{\Delta_4}.$$

That (32) is a solution of (31) may be seen by direct substitution. If there were two distinct solutions of (31), the non-vanishing difference of those solutions would satisfy (28), the homogeneous equation. In view of the inequality of Theorem 15 and the necessary condition in Theorem 14, this is impossible.

Equation (32) is readily altered to fit the case where the homogeneous equation also has a solution. Let us suppose, for example, that in (31) we find $\Delta_2 = \Delta_4 = 0$, $\Delta_1 \Delta_3 \neq 0$. Then there exists no solution of (31) unless $P = 0$ and $W = 0$. If these conditions are satisfied, the general solution of (31) is

$$y = \frac{I}{\Delta_1} + P_1 + i \frac{Q}{\Delta_3} + W_1,$$

where P_1 is any pseudo-invariant, W_1 any i^3 -variant and the other symbols are as in Theorem 15. There is here a noticeable resemblance to the general solution of a non-homogeneous ordinary linear differential equation.

If in Theorem 15 we use the representation of Theorem 13, instead of that of Theorem 12, we need only to replace (31) by

$$(31') \quad a_3 \sigma^3 y + a_2 \sigma^2 y + a_1 \sigma y + a_0 y = I_1 + P_1 + x(I_2 + P_2) + D(I_3 + P_3),$$

and (32) by

$$(32') \quad y = \frac{I_1}{\Delta_1} - \frac{P_1}{\Delta_2} - \frac{a_3 - a_1}{\Delta_3 \Delta_4} [D(I_2 - P_2) - x(I_3 - P_3)] \\ - \frac{a_2 - a_0}{\Delta_3 \Delta_4} [x(I_2 + P_2) + D(I_3 + P_3)].$$

9. Linear differential equations invariant under σ . We have incidentally solved the problem of determining what linear differential equations are invariant under σ . Theorem 14 and the remarks following it yield at once

THEOREM 16. *Let y be a linear differential operator of type P and let F be an undetermined function of x . Then a necessary and sufficient condition that $yF = 0$ be invariant under σ is that y be a t -variant with respect to σ .*

This is an interpretation of the fact that for $\sigma y = cy$ to have a solution for constant c it is necessary and sufficient that $c^4 = 1$.

By means of the representation (18) of Theorem 13 we are able to state this result in another form sometimes more useful.

THEOREM 17. *Let y be a linear differential operator of type P and let F be an undetermined function of x . Then a necessary and sufficient condition that $yF = 0$ be invariant under σ is that y be expressible in one of the four forms (33)–(36);*

$$(33) \quad y = I_1,$$

$$(34) \quad y = P_1,$$

$$(35) \quad y = x(I_2 + P_2) + iD(I_2 - P_2),$$

$$(36) \quad y = x(I_2 + P_2) - iD(I_2 - P_2),$$

where I_1, I_2 are invariants, P_1, P_2 are pseudo-invariants, and $i = \sqrt{-1}$.

10. Linear operational equations in σ : variable coefficients. We may generalize equation (31) to the case where the coefficients are themselves linear differential operators. We consider

$$(37) \quad a_3(\sigma^3 y)b_3 + a_2(\sigma^2 y)b_2 + a_1(\sigma y)b_1 + a_0 y b_0 = A,$$

where the a_i, b_i ; $i = 0, 1, 2, 3$, and A are known linear differential operators and y is to be determined. We shall illustrate our two methods of attack on (37) by means of a numerical example. Consider

$$(38) \quad x\sigma y - Dy = 0.$$

If we use the representation $y = I + P + Q + W$ as indicated in Theorem 12, we find that

$$(39) \quad x(I - P) + ix(Q - W) - D(I + P) - D(Q + W) = 0.$$

Operating on (39) with $\sigma, \sigma^2, \sigma^3$, and combining the resulting equations, we readily obtain $I = 0, P = 0$, and

$$(40) \quad (D - ix)Q + (D + ix)W = 0.$$

Further, if we substitute $y = Q + W$ into (38) we get (40). Hence, the general solution of (38) is $y = Q + W$ where Q and W are respectively any i -variant and any i^3 -variant subject to the restriction (40).

Let us now attack (38) with the representation made available by Theorem 13. In this case the solution appears in a more satisfactory form. Let $y = I_1 + P_1 + x(I_2 + P_2) + D(I_3 + P_3)$. Then from (38) we get

$$(41) \quad (x-D)I_1 - (x+D)P_1 \\ - (2xD+1)I_2 - P_2 + x^2(I_3 - P_3) - D^2(I_3 + P_3) = 0.$$

Using σ on (41) we arrive at

$$(42) \quad -(x+D)I_1 - (D-x)P_1 \\ + (2xD+1)I_2 + P_2 + D^2(I_3 + P_3) - x^2(I_3 - P_3) = 0.$$

Equations (41) and (42) combine to yield $2D(I_1 + P_1) = 0$, hence $I_1 = P_1 = 0$. We return to (41) which has become

$$(43) \quad (2xD+1)I_2 + P_2 - x^2(I_3 - P_3) + D^2(I_3 + P_3) = 0.$$

Substituting for P_2 from (43) into the assumed expression for y we get

$$y = -2x^2DI_2 - (xD^2 - D - x^3)I_3 - (xD^2 - D + x^3)P_3.$$

Since I_2 and I_3 may be any invariants and P_3 any pseudo-invariant, we shall write

$$(44) \quad y = x^2DI_4 + (xD^2 - D - x^3)I_5 + (xD^2 - D + x^3)P_5.$$

Then

$$x\sigma y = x(D^2x)I_4 + x(-Dx^2 - x + D^3)I_5 - x(-Dx^2 - x - D^3)P_5 \\ = x(xD^2 + 2D)I_4 + x(D^3 - x^2D - 3x)I_5 + x(D^3 + x^2D + 3x)P_5,$$

and

$$Dy = (x^2D^2 + 2xD)I_4 + (xD^3 - x^3D - 3x^2)I_5 + (xD^3 + x^3D + 3x^2)P_5.$$

Thus we have the result: the general solution of (38) is (44), where I_4 and I_5 are any invariants and P_5 is any pseudo-invariant with respect to σ . In this case the representation in Theorem 13 is seen to have a considerable advantage over that in Theorem 12.

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